"The Bending of Electric Waves round a Conducting Obstacle: Amended Result." By H. M. MACDONALD, M.A., F.R.S.

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I have recently (May 3) received an intimation from the Secretaries of the Royal Society that Lord Rayleigh has questioned the validity of my analysis* of the problem of bending of electric waves round a conducting obstacle, the ground of the criticism being that the shortness of the wave-length involves that the important harmonics in the expansion are of high order comparable with the ratio of the circumference of the sphere to the wave-length, and that for them the approximations in the paper are not valid. Subsequently I have learned that M. Poincaré has made a similar objection.

I have at once to admit the validity of the objection thus raised. As it appears that it is still to some degree uncertain what phenomena would be indicated by theory, I venture to submit the following correction and development of my analysis, which gives a solution of the problem agreeing with Lord Rayleigh's anticipation (p. 40, *supra*).

Starting from the expression for $\partial\psi/\partial\mu_{r=a}$, given on p. 254,

$$\frac{\partial\psi}{\partial\mu_{r=a}} = \frac{\partial}{\partial\mu}\sum_1^\infty g_n(r_1)\left[a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a) - \frac{a^{\frac{1}{2}}K_{n+1}(\iota\kappa a)}{\frac{\partial}{\partial a}\{a^{\frac{1}{2}}K_{n+\frac{1}{2}}(\iota\kappa a)\}}\frac{\partial}{\partial a}\{a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a)\}\right](1-\mu^2)\frac{\partial P_n}{\partial\mu},$$

it may be written

$$\frac{\partial\psi}{\partial\mu_{r=a}} = \frac{\partial}{\partial\mu}\sum^\infty g_n(r_1)\left[a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a) + \frac{1}{\iota\kappa}\frac{\partial}{\partial a}\{a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a)\}\right](1-\mu^2)\frac{\partial P_n}{\partial\mu}$$

$$-\frac{\partial}{\partial\mu}\sum_1^\infty g_n(r_1)\left[\frac{a^{\frac{1}{2}}K_{n+\frac{1}{2}}(\iota\kappa a)}{\frac{\partial}{\partial a}\{a^{\frac{1}{2}}K_{n+\frac{1}{2}}(\iota\kappa a)} + \frac{1}{\iota\kappa}\right]\frac{\partial}{\partial a}\{a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a)\}(1-\mu^2)\frac{\partial P_n}{\partial\mu};$$

that is

$$\frac{\partial\psi}{\partial\mu_{r=a}} = \frac{\partial}{\partial\mu}\left[f(a) + \frac{1}{\iota\kappa}\frac{\partial}{\partial a}\{f(a)\}\right] - S,$$

where

$$S = \frac{\partial}{\partial\mu}\sum_1^\infty g_n(r_1)\left[\frac{a^{\frac{1}{2}}K_{n+\frac{1}{2}}(\iota\kappa a)}{\frac{\partial}{\partial a}\{a^{\frac{1}{2}}K_{n+\frac{1}{2}}(\iota\kappa a)} + \frac{1}{\iota\kappa}\right]\frac{\partial}{\partial a}\{a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a))\}(1-\mu^2)\frac{\partial P_n}{\partial\mu_1}.$$

Writing

$$\kappa a = z, \quad z^{\frac{1}{2}}J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,u_n(z), \quad z^{\frac{1}{2}}J_{-n-\frac{1}{2}}(z) = (-)^n\sqrt{\frac{2}{\pi}}\,v_n(z),$$

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* 'Roy. Soc. Proc.,' vol. 71, p. 251.

then

$$z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota z) = \frac{\pi e^{(-n+\frac{1}{2})\frac{\pi\iota}{2}}}{2\sin(n+\frac{1}{2})\pi}\left[z^{\frac{1}{2}}\mathrm{J}_{-n-\frac{1}{2}}(z) - e^{(n+\frac{1}{2})\pi\iota}z^{\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(z)\right],$$

that is

$$z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota z) = \sqrt{\frac{\pi}{2}}\,e^{-(n+\frac{1}{2})\pi\iota}\{v_n(z) - \iota u_n(z)\},$$

$$\frac{\partial}{\partial z}\{z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota z)\} = \sqrt{\frac{\pi}{2}}\,e^{-(n+\frac{1}{2})\pi\iota}\{v_n'(z) - \iota u_n'(z)\};$$

and putting

$$u_n(z) = \mathrm{R}_n^{\frac{1}{2}}(z)\sin\phi_n, \qquad v_n(z) = \mathrm{R}_n^{\frac{1}{2}}(z)\cos\phi_n,$$

then

$$v_n'(z) - \iota u_n'(z) = \frac{1}{2}\frac{\mathrm{R}_n'(z)}{\mathrm{R}_n^{\frac{1}{2}}(z)}e^{-\iota\phi_n} - \iota\mathrm{R}_n^{\frac{1}{2}}(z)\frac{\partial\phi_n}{\partial z}e^{-\iota\phi_n}.$$

Now

$$v_n(z)u_n'(z) - v_n'(z)u_n(z) = 1,$$

whence

$$\mathrm{R}_n(z)\frac{\partial\phi_n}{\partial z} = 1,$$

therefore

$$v_n'(z) - \iota u_n'(z) = \{\tfrac{1}{2}\mathrm{R}_n'(z) - \iota\}\frac{e^{-\iota\phi_n}}{\mathrm{R}_n^{\frac{1}{2}}(z)};$$

hence

$$\frac{a^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)}{\frac{\partial}{\partial a}\left\{a^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)\right.} + \frac{1}{\iota\kappa} = \left\{\frac{\mathrm{R}_n(z)}{\frac{1}{2}\mathrm{R}_n'(z) - \iota} + \frac{1}{\iota}\right\}\frac{1}{\kappa},$$

or

$$\frac{a^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)}{\frac{\partial}{\partial a}\{a^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)\}} + \frac{1}{\iota\kappa} = \frac{1}{\iota\kappa}\{1 - \mathrm{R}_n(z)\cos\chi_n e^{\iota\chi_n}\},$$

where

$$\tan\chi_n + \tfrac{1}{2}\mathrm{R}_n'(z) = 0.$$

Again

$$\frac{\partial}{\partial a}\{a^{\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa a)\} = \sqrt{\frac{2\kappa}{\pi}}\,u_n'(z),$$

that is

$$\frac{\partial}{\partial a}\{a^{\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa a)\} = \sqrt{\frac{2\kappa}{\pi}}\left[\frac{1}{2}\frac{\mathrm{R}_n'(z)}{\mathrm{R}_n^{\frac{1}{2}}(z)}\sin\phi_n + \mathrm{R}_n^{\frac{1}{2}}(z)\cos\phi_n\frac{\partial\phi_n}{\partial z}\right],$$

or

$$\frac{\partial}{\partial a}\{a^{\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa a)\} = \sqrt{\frac{2\kappa}{\pi}}\frac{\cos(\phi_n + \chi_n)}{\mathrm{R}_n^{\frac{1}{2}}(z)\cos\chi_n}.$$

Further

$$g_n(r_1) = -\kappa r_1^{-\frac{1}{2}}e^{\pi\iota/4}\{e^{(n-1)\frac{\pi\iota}{2}}\mathrm{K}_{n-\frac{1}{2}}(\iota\kappa r_1) + e^{(n+1)\frac{\pi\iota}{2}}\mathrm{K}_{n+\frac{3}{2}}(\iota\kappa r_1)\},\text{*}$$

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* 'Roy. Soc. Proc.,' vol. 71, p. 253.

whence

$$g_n(r_1) = -\kappa r_1{}^{-\frac{1}{2}} e^{(n+\frac{3}{2})\frac{\pi\iota}{2}} \frac{2n+1}{\iota\kappa r_1} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa r_1),$$

that is

$$g_n(r_1) = \frac{2n+1}{\kappa^{\frac{1}{2}} r_1{}^2} e^{(n+\frac{5}{2})\frac{1}{2}\pi\iota} \sqrt{\frac{\pi}{2}}\, e^{-(n+\frac{1}{2})\frac{1}{2}\pi\iota} \{v_n(\kappa r_1) - \iota u_n(\kappa r_1)\},$$

or

$$g_n(r_1) = -\frac{2n+1}{\kappa^{\frac{1}{2}} r_1{}^2} \sqrt{\frac{\pi}{2}} \{v_n(\kappa r_1) - \iota u_n(\kappa r_1)\}.$$

Hence

$$\mathrm{S} = -\frac{\partial}{\partial\mu} \sum_1^\infty \frac{2n+1}{\kappa^{\frac{1}{2}} r_1{}^2} \sqrt{\frac{\pi}{2}} \{v_n(\kappa r_1) - \iota u_n(\kappa r_1)\} \frac{1}{\iota\kappa} \{1 - \mathrm{R}_n(z) \cos\chi_n e^{\iota\chi_n}\}$$
$$\sqrt{\frac{2\kappa}{\pi}} \frac{\cos(\phi_n + \chi_n)}{\mathrm{R}_n{}^{\frac{1}{2}}(z) \cos\chi_n} (1-\mu^2) \frac{\partial \mathrm{P}_n}{\partial\mu},$$

that is

$$\mathrm{S} = \frac{\iota}{\kappa r_1{}^2} \frac{\partial}{\partial\mu} \sum_1^\infty (2n+1) \{v_n(\kappa r_1) - \iota u_n(\kappa r_1)\} \{1 - \mathrm{R}_n(z) \cos\chi_n e^{\iota\chi_n}\}$$
$$\frac{\cos(\phi_n + \chi_n)}{\mathrm{R}_n{}^{\frac{1}{2}}(z) \cos\chi_n} (1-\mu^2) \frac{\partial \mathrm{P}_n}{\partial\mu},$$

or

$$\mathrm{S} = -\frac{\iota}{\kappa r_1{}^2} \sum_1^\infty n(n+1)(2n+1) \{v_n(\kappa r_1) - \iota u_n(\kappa r_1)\} \{1 - \mathrm{R}_n(z) \cos\chi_n e^{\iota\chi_n}\}$$
$$\frac{\cos(\phi_n + \chi_n)}{\mathrm{R}_n{}^{\frac{1}{2}}(z) \cos\chi_n} \mathrm{P}_n.$$

Now for values of θ ($= \cos^{-1}\mu$) which are not near to 0 or π, P_n may be replaced by $\sqrt{\frac{2}{\pi n}} \cos\left\{(n+\frac{1}{2})\theta - \frac{\pi}{4}\right\}$ for large values of n; further, for values of n small compared with z, neglecting small quantities, χ_n is zero and $\mathrm{R}_n(z)$ is unity. Hence

$$\mathrm{S} = -\frac{\iota}{\kappa r_1{}^2} \sum_{n_0}^\infty n(n+1)(2n+1) \{v_n(\kappa r_1) - \iota u_n(\kappa r_1)\} \{1 - \mathrm{R}_n(z) \cos\chi_n e^{\iota\chi_n}\}$$
$$\frac{\cos(\phi_n + \chi_n)}{\mathrm{R}_n{}^{\frac{1}{2}}(z) \cos\chi_n} \sqrt{\frac{2}{\pi n \sin\theta}} \cos\left\{(n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\},$$

where n_0 is large, but not comparable with z. Since θ is not small, κr_1 may be replaced by z, and then

$$\mathrm{S} = -\frac{\iota}{\kappa a^2} \sum_{n_0}^\infty n(n+1)(2n+1) \sqrt{\frac{2}{\pi n \sin\theta}} \{1 - \mathrm{R}_n(z) \cos\chi_n e^{\iota\chi_n}\}$$
$$\frac{\cos(\phi_n + \chi_n)}{\cos\chi_n} e^{-\iota\phi_n} \cos\left\{(n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\},$$

that is

$$S = -\frac{\iota}{2\kappa a^2}\sum_{n_0}^{\infty} n(n+1)(2n+1)\sqrt{\frac{2}{\pi n\sin\theta}}\,\{1 - R_n(z)\cos\chi_n e^{\iota\chi_n}\}$$

$$\frac{e^{-\iota\phi_n}}{\cos\chi_n}\left[\cos\left\{\phi_n+\chi_n+(n+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right\}+\cos\left\{\phi_n+\chi_n-(n+\tfrac{1}{2})\,\theta+\frac{\pi}{4}\right\}\right]$$

Writing

$$\frac{\iota R_n(z)}{4\kappa a^2}\sqrt{\frac{2}{\pi n\sin\theta}}\,n(n+1)(2n+1) = f_1(n),$$

$$\frac{\iota}{4\kappa a^2\cos\chi_n}\sqrt{\frac{2}{\pi n\sin\theta}}\,n(n+1)(2n+1) = f_2(n),$$

the above becomes

$$S = S_1+S_2+S_3+S_4-S_5-S_6-S_7-S_8,$$

where

$$S_1 = \sum_{n_0}^{\infty} f_1(n)\,e^{\iota\left\{2\chi_n+(n+\frac{1}{2})\theta-\frac{\pi}{4}\right\}}, \qquad S_5 = \sum_{n_0}^{\infty} f_2(n)e^{\iota\left\{\chi_n+(n+\frac{1}{2})\theta-\frac{\pi}{4}\right\}},$$

$$S_2 = \sum_{n_0}^{\infty} f_1(n)\,e^{\iota\left\{2\chi_n-(n+\frac{1}{2})\theta+\frac{\pi}{4}\right\}}, \qquad S_6 = \sum_{n_0}^{\infty} f_2(n)\,e^{\iota\left\{\chi_n-(n+\frac{1}{2})\theta+\frac{\pi}{4}\right\}},$$

$$S_3 = \sum_{n_0}^{\infty} f_1(n)\,e^{-\iota\left\{2\phi_n+(n+\frac{1}{2})\theta-\frac{\pi}{4}\right\}}, \qquad S_7 = \sum_{n_0}^{\infty} f_2(n)\,e^{-\iota\left\{2\phi_n+\chi_n+(n+\frac{1}{2})\theta-\frac{\pi}{4}\right\}},$$

$$S_4 = \sum_{n_0}^{\infty} f_1(n)\,e^{-\iota\left\{2\phi_n-(n+\frac{1}{2})\theta+\frac{\pi}{4}\right\}}. \qquad S_8 = \sum_{n_0}^{\infty} f_2(n)\,e^{-\iota\left\{2\phi_n+\chi_n-(n+\frac{1}{2})\theta+\frac{\pi}{4}\right\}}.$$

If any value of n_1 of n be chosen, and n is written $n_1+\nu$, the quantities χ_n, ϕ_n can be expressed in the form $A_0 + A_1\frac{\nu}{z} + A_2\frac{\nu^2}{z^2} + \ldots$ for values of ν for which the series converge; similarly for $f_1(n)$, $f_2(n)$. The group of terms in any of the series S_1 etc., for which this holds for n_1, will contribute nothing to the result unless the coefficient of ν in the exponent vanishes; also when the coefficient of ν in the exponent vanishes and the coefficient of ν^2 does not, the sum of the corresponding group of terms

$$\Sigma\left(A_0 + A_1\frac{\nu}{2} + \ldots\right)e^{\iota\left(B_0 z + B_2\frac{\nu^2}{z} + \ldots\right)}$$

is

$$A_0\sqrt{\frac{\pi z}{|B_2|}}\,e^{\iota\left(B_0 z \pm \frac{\pi}{4}\right)},$$

the upper or lower sign being taken according as B_2 is positive or negative.*

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* Lorenz, 'Œuvres Scientifiques,' vol. 1, p. 425.

Now it may be shown that

$$\mathrm{R}_n(z) = \sum_{s=0}^{\infty} \frac{\Pi(n+s)\,\Pi(s-\frac{1}{2})}{\Pi(n-s)\,\Pi(s)\,\Pi(-\frac{1}{2})}\,\frac{1}{z^{2s}},$$

and hence that

$$\mathrm{R}_n(z) = \frac{z}{\sqrt{z^2-(n+\frac{1}{2})^2}},$$

as long as $z > n+\frac{1}{2}$ and $z-(n+\frac{1}{2})$ is of higher order than $z^{\frac{1}{3}}$; also from the relation

$$\mathrm{R}_n \frac{\partial\phi_n}{\partial z} = 1,$$

it follows that

$$\phi_n = \sqrt{z^2-(n+\tfrac{1}{2})^2} - \frac{n\pi}{2} + (n+\tfrac{1}{2})\sin^{-1}\frac{n+\frac{1}{2}}{z},$$

subject to the same limitations. Further, when $z-(n+\frac{1}{2})$ is of a lower order than $z^{\frac{1}{3}}$,

$$\mathrm{R}_n(z) = \frac{(n+\frac{1}{2})^{\frac{1}{3}}}{3^{\frac{5}{6}}\sqrt{\pi}}\left[\Pi(-\tfrac{5}{6}) + \Pi(-\tfrac{1}{2})(n+\tfrac{1}{2}-z)\left(\frac{24}{n+\frac{1}{2}}\right)^{\frac{1}{3}}\right.$$
$$\left. + \Pi(-\tfrac{1}{6})(n+\tfrac{1}{2}-z)^2\left(\frac{24}{n+\frac{1}{2}}\right)^{\frac{2}{3}}\tfrac{1}{2} + \ldots\right],$$

$$\phi_n = \frac{\pi}{6} + \mathrm{R}_n(z-n-\tfrac{1}{2}) - \frac{\mathrm{R}'_n}{\mathrm{R}_n{}^2}\,\frac{(z-n-\frac{1}{2})^2}{2} + \ldots,$$

where R_n, R'_n, ... denote the values of $\mathrm{R}_n(z)$, $\mathrm{R}'_n(z)$. ... when $z = n+\frac{1}{2}$. When $n+\frac{1}{2} > z$, writing

$$u_n(z) = \mathrm{T}_n(z)\,e^{\tau_n}, \qquad v_n(z) = \mathrm{T}_n(z)\,e^{-\tau_n},$$

$$2\mathrm{T}_n(z) = \frac{z}{\sqrt{(n+\frac{1}{2})^2-z^2}},$$

as long as $n+\frac{1}{2} > z$ and $n+\frac{1}{2}-z$ is of a higher order than $z^{\frac{1}{3}}$; also

$$\tau_n = -\tfrac{1}{2}\log 2 + (n+\tfrac{1}{2})\log\frac{n+\frac{1}{2}-\sqrt{(n+\frac{1}{2})^2-z^2}}{z} + \sqrt{(n+\tfrac{1}{2})^2-z^2};$$

and when $n+\frac{1}{2}-z$ is of a lower order than $z^{\frac{1}{3}}$,

$$2\mathrm{T}_n(z) = \frac{z^{\frac{1}{3}}}{3^{\frac{5}{6}}\sqrt{\pi}}\left[\Pi(-\tfrac{5}{6})\sin\frac{\pi}{3} + \Pi(-\tfrac{1}{2})\sin\frac{3\pi}{3}(n+\tfrac{1}{2}-2)\left(\frac{24}{2}\right)^{\frac{1}{3}}\right.$$
$$\left. + \Pi(-\tfrac{1}{6})\sin\frac{5\pi}{3}(n+\tfrac{1}{2}-z)^2\left(\frac{24}{z}\right)^{\frac{2}{3}}\tfrac{1}{2} + \ldots\right]$$

$$\tau_n = -\tfrac{1}{4}\log 3 + \frac{1}{2\mathrm{T}_n}(z-n-\tfrac{1}{2}) - \frac{\mathrm{T}_n{}'}{2\mathrm{T}_n{}^2}(z-n-\tfrac{1}{2})^2\tfrac{1}{2} + \ldots,$$

where T_n, T_n', . . . denote the values of $T_n(z)$, $T_n'(z)$, . . ., when $z = n + \frac{1}{2}$.*

From the expression for $R_n(z)$ it follows that

$$\tfrac{1}{2}R'_n(z) = -\sum_{s=1}^{\infty} \frac{\Pi(n+s)\,\Pi(s-\frac{1}{2})}{\Pi(n-s)\,\Pi(s-1)\,\Pi(-\frac{1}{2})}\,\frac{1}{z^{2s+1}},$$

hence $\tan\chi_n$ is positive and increases with n, thus χ_n lies between 0 and $\frac{1}{2}\pi$ and $\partial\chi_n/\partial n$ is positive; therefore S_1 and S_5 both vanish. When $z-(n+\frac{1}{2})$ is of higher order than $z^{\frac{1}{3}}$

$$R_n'(z) = \frac{-(n+\frac{1}{2})^2}{\{z^2-(n+\frac{1}{2})^2\}^{\frac{3}{2}}},$$

hence

$$\tan\chi_n = \tfrac{1}{2}\,\frac{(n+\frac{1}{2})^2}{\{z^2-(n+\frac{1}{2})^2\}^{\frac{3}{2}}},$$

and

$$\sec^2\chi_n\,\frac{\partial\chi_n}{\partial n} = \frac{n+\frac{1}{2}}{\{z^2-(n+\frac{1}{2})^2\}^{\frac{3}{2}}} + \frac{3}{2}\,\frac{(n+\frac{1}{2})^3}{\{z^2-(n+\frac{1}{2})^2\}^{\frac{5}{2}}},$$

therefore as n increases, subject to the limitation that $z-(n+\frac{1}{2})$ is of higher order than $z^{\frac{1}{3}}$, $R_n(z)$ tends to become of the order $(n+\frac{1}{2})^{\frac{1}{3}}$, $R_n'(z)$ of the order unity, and $\partial\chi_n/\partial n$ of the order $(n+\frac{1}{2})^{-\frac{1}{3}}$. When $z-(n+\frac{1}{2})$ is of lower order than $z^{\frac{1}{3}}$, it follows from the relation

$$R_n(z) = \frac{(n+\frac{1}{2})^{\frac{1}{3}}}{3^{\frac{5}{6}}\sqrt{\pi}}\left[\Pi(-\tfrac{1}{6}) + \Pi(-\tfrac{1}{2})\,(n+\tfrac{1}{2}-z)\left(\frac{24}{n+\frac{1}{2}}\right)^{\frac{1}{3}} + \ldots\right]$$

that $R_n(z)$ is of the order $(n+\frac{1}{2})^{\frac{1}{3}}$, $R'_n(z)$ of the order unity, and $\partial\chi_n/\partial n$ of the order $(n+\frac{1}{2})^{-\frac{1}{3}}$. These latter conditions also hold when $n+\frac{1}{2}-z$ is of lower order than $z^{\frac{1}{3}}$. When $n+\frac{1}{2}-z$ is of higher order than $z^{\frac{1}{3}}$, $R_n(z)$ is given by

$$R_n(z) = 2T_n(z)\cosh 2\tau_n,$$

and

$$R_n'(z) = 2T_n'(z)\cosh 2\tau_n + 4T_n(z)\sinh 2\tau_n\,\frac{\partial\tau_n}{\partial z},$$

that is

$$R_n'(z) = 2T_n'(z)\cosh 2\tau_n + 2\sinh 2\tau_n,$$

whence

$$-2\sec^2\chi_n\,\frac{\partial\chi_n}{\partial n} = 2\,\frac{\partial T_n'(z)}{\partial n}\cosh 2\tau_n + 4T_n'(z)\sinh 2\tau_n\,\frac{\partial\tau_n}{\partial n} + 4\cosh 2T_n\,\frac{\partial\tau_n}{\partial n};$$

now

$$2T_n'(z) = \frac{(n+\frac{1}{2})^2}{\{(n+\frac{1}{2})^2-z^2\}^{3}},$$

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* These results are given by Lorenz, 'Œuvres Scientifiques,' vol. 1, pp. 435—479.

therefore $2\dfrac{\partial \mathrm{T}_n'(z)}{\partial n}$ is negative and at most of the order $(n+\frac{1}{2})^{-\frac{1}{3}}$, $\mathrm{T}_n'(z)$ being at most of the order unity; further, when $z = n+\frac{1}{2}$, $\tau_n = -\frac{1}{4}\log 3$, and as n increases T_n diminishes rapidly, hence $\cosh 2\tau_n \cos^2\chi_n$ and $\sinh 2\tau_n \sin^2\chi_n$ are less than unity and diminish rapidly as n increases; also writing $n+\frac{1}{2} = z\cosh\delta$,

$$\tau_n = -\tfrac{1}{2}\log 2 - z\delta\cosh\delta + z\sin\delta,$$

$$\frac{\partial \tau_n}{\partial n} = -\delta,$$

hence, unless δ is small, τ_n is a large negative quantity, and therefore $\cosh 2\tau_n \cos^2\chi_n$, $\sinh 2\tau_n \cos^2\chi_n$, by the above, very small, whence it follows that $\partial\chi_n/\partial n$ is always very small. The series S_2 and S_6 therefore both vanish.

It is known that neither $u_n(z)$ or $v_n(z)$ can vanish for a value of n, which satisfies the condition that $n + \frac{1}{2} > \dfrac{2z}{\pi}$, and the last time $v_n(z)$ vanishes as n increases the value of ϕ_n is $\frac{1}{2}\pi$; as n increases farther ϕ_n diminishes to zero. When $z-(n+\frac{1}{2})$ is of higher order than $z^{\frac{1}{3}}$, writing $n+\frac{1}{2} = z\sin\alpha$, where $\frac{1}{2}\pi > \alpha > 0$,

$$\phi_n = z\cos\alpha - \tfrac{1}{2}n\pi + (n+\tfrac{1}{2})\alpha,$$

$$\frac{\partial\phi_n}{\partial n} = \alpha - \tfrac{1}{2}\pi,\quad \frac{\partial^2\phi_n}{\partial n^2} = \frac{1}{z\cos\alpha},$$

hence $\dfrac{\partial\phi_n}{\partial n}$ is negative, and as n increases $-\dfrac{\partial\phi_n}{\partial n}$ diminishes and tends to the order $(n+\frac{1}{2})^{-\frac{1}{3}}$. When $z-(n+\frac{1}{2})$ is of lower order than $z^{\frac{1}{3}}$, $\dfrac{\partial\phi_n}{\partial n}$ is given by

$$\frac{\partial\phi_n}{\partial n} = -\frac{1}{\mathrm{R}_n} + (z-n-\tfrac{1}{2})\frac{\mathrm{R}'_n}{\mathrm{R}_n^2} + \ldots,$$

whence, remembering that R'_n is negative and R_n is of the order $(n+\frac{1}{2})^{\frac{1}{3}}$, $\dfrac{\partial\phi_n}{\partial n}$ is negative and of the order $(n+\frac{1}{2})^{-\frac{1}{3}}$. When $n+\frac{1}{2} > z$, using the relation $\tan\phi_n = e^{2\tau_n}$,

$$\sec^2\phi_n\frac{\partial\phi_n}{\partial n} = 2e^{2\tau_n}\frac{\partial\tau_n}{\partial n},$$

and therefore

$$\frac{\partial\phi_n}{\partial n} = -\frac{2\delta e^{4\tau_n}}{1+e^{4\tau_n}},$$

where

$$z\cosh\delta = n+\tfrac{1}{2},$$

hence $\frac{\partial\phi_n}{\partial n}$ is negative, and $-\frac{\partial\phi_n}{\partial n}$ diminishes as the absolute value of τ_n increases, therefore $-\frac{\partial\phi_n}{\partial n}$ is, when $n+\frac{1}{2}>z$, at most of an order $(n+\frac{1}{2})^{-\frac{1}{3}}$. Since $\frac{\partial\phi_n}{\partial n}$ is always negative the series S_1 vanishes, and since $\frac{\partial\phi_n}{\partial n}$ is always negative, and $\frac{\partial\chi_n}{\partial n}$ is always very small, the series S_8 vanishes.

It remains to evaluate the series S_3 and S_7. Writing $n = n_1+\nu$, the exponent in the series S_3 becomes

$$-\iota\left(2\phi_{n_1}+2\frac{\partial\phi_n}{\partial n_1}\nu+\frac{\partial^2\phi_n}{\partial n_1^2}\nu^2+\ \ldots\ +(n_1+\tfrac{1}{2})+\nu\theta-\frac{\pi}{4}\right),$$

and the coefficient of ν in this vanishes, if

$$2\frac{\partial\phi_n}{\partial n_1}+\theta = 0,^*$$

and putting $n_1+\frac{1}{2} = z\sin\alpha$, this becomes

$$2\alpha-\pi+\theta = 0,$$

and the sum of the corresponding group of terms in the series S_3 is

$$f_1(n_1)\sqrt{\frac{\pi}{\frac{\partial^2\phi_n}{\partial n_1^2}}}\,e^{-\iota\left\{2\phi_{n_1}+(n_1+\frac{1}{2})\theta\right\}};$$

now

$$\phi_{n_1} = z\cos\alpha-\tfrac{1}{2}n_1\pi+(n_1+\tfrac{1}{2})\,\alpha,$$

that is

$$2\phi_{n_1}+(n_1+\tfrac{1}{2})\theta = 2z\cos\alpha-n_1\pi+(2n_1+1)\,\alpha+(n_1+\tfrac{1}{2})\,\theta,$$

whence

$$2\phi_{n_1}+(n_1+\tfrac{1}{2})\,\theta = 2z\cos\alpha+\tfrac{1}{2}\pi,$$

and the sum of the group of terms, therefore, is

$$f_1(n_1)\sqrt{\pi z\cos\alpha}\,e^{-\iota(2z\cos\alpha+\frac{1}{2}\pi)},$$

that is, since, neglecting all but the terms of highest order,

$$f_1(n_1) = \frac{\iota}{4\kappa a^2\cos\alpha}\sqrt{\frac{2}{\pi z\sin\alpha\sin\theta}}\,2z^3\sin^3\alpha,$$

$$S_3 = \frac{\kappa^2 a\cos^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}\,e^{-2\iota\kappa a\sin\frac{1}{2}\theta}$$

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[*] This equation cannot be satisfied, by the preceding, for a value n_1 of n, which is not such that $z-(n+\frac{1}{2})$ is positive and of higher order than $z^{\frac{1}{3}}$.

Similarly the value of n, for which a group of terms of S_7 has a value different from zero, is given by

$$2\frac{\partial \phi_n}{\partial n} + \frac{\partial \chi_n}{\partial n} + \theta = 0,$$

and remembering that $\frac{\partial \chi_n}{\partial n}$ is always very small, the corresponding value of n is n_1 as in the previous case. Now, when $n_1 + \frac{1}{2} = z \sin \alpha$,

$$\tan \chi_n \doteq \frac{\sin^2 \alpha}{z \cos^3 \alpha},$$

and is, therefore, to the order required, zero, $\cos \chi_n$ being unity, hence

$$S_7 = \tfrac{1}{2}\kappa^2 a \cos^2 \tfrac{1}{2}\theta \, e^{-2\iota\kappa a \sin \frac{1}{2}\theta},$$

therefore

$$S = \frac{\kappa^2 a \cos^2 \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta} (1 - \sin \tfrac{1}{2}\theta).$$

The value of $\frac{\partial \psi}{\partial \mu_{r=a}}$ is, therefore, for values of θ which are not near to 0 or π, given by

$$\frac{\partial \psi}{\partial \mu_{r=a}} = \frac{\partial}{\partial \mu}\left[f(a) + \frac{1}{\iota\kappa}\frac{\partial}{\partial a}\{f(a)\}\right] - \frac{\kappa' a \cos^2 \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta} (1 - \sin \tfrac{1}{2}\theta),$$

the terms of highest order only being retained. Now

$$f(a) = \frac{a^2(1-\mu^2)}{R_0}\frac{\partial}{\partial R_0}\frac{e^{-\iota\kappa R_0}}{R_0},*$$

where

$$R_0{}^2 = a^2 + r_1{}^2 - 2ar_1\mu,$$

that is,

$$f(a) = -\iota\kappa\frac{a^2(1-\mu^2)}{R_0{}^2} e^{-\iota\kappa R_0},$$

to the order adopted, and

$$\frac{\partial}{\partial \mu}\left[f(a) + \frac{1}{\iota\kappa}\frac{\partial}{\partial a}\{f(a)\}\right] = -\frac{\partial}{\partial \mu}\frac{\iota\kappa a^2(1-\mu^2)}{R_0{}^2}\left(1 - \frac{a - r_1\mu}{R_0}\right) e^{-\iota\kappa R_0}$$

or

$$\frac{\partial}{\partial \mu}\left[f(a) + \frac{1}{\iota\kappa}\frac{\partial}{\partial a}\{f(a)\}\right] = \frac{\kappa^2 a^3 r_1}{R_0{}^3}(1-\mu^2)\left(1 - \frac{a - r_1\mu}{R_0}\right) e^{-\iota\kappa R_0},$$

which for values of θ not near to zero, becomes on putting $r_1 = a$,

$$\frac{\partial}{\partial \mu}\left[f(a) + \frac{1}{\iota\kappa}\frac{\partial}{\partial a}\{f(a)\}\right] = \frac{\kappa^2 a \cos^2 \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta}(1 - \sin \tfrac{1}{2}\theta).$$

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* 'Roy. Soc. Proc.,' vol. 71, p. 254.

Therefore, when θ is not small, $\frac{\partial\psi}{\partial\mu}_{r=a}$ vanishes to the first order at least, so that there is no first order effect at a point on the surface of the sphere which is at a finite angular distance from the oscillator.

"On the Structure of Gold Leaf and the Absorption Spectrum of Gold." By J. W. MALLET, F.R.S., Professor of Chemistry in the University of Virginia. Received May 22,—Read June 11, 1903.

(Abstract.)

Attention is drawn to numerous irregularly distributed black lines which are to be seen in gold leaf examined with the microscope by transmitted light. These lines are shown to depend on the presence of minute wires or threads of the metal, unconnected with its crystalline structure, but produced in the process of gold beating by the stretching, along lines of weakness, of the animal membrane between sheets of which the gold is placed, thus developing minute trough-like wrinkles into which the soft metal is forced.

The results are given of an examination of the absorption spectrum—in the visible, ultra-violet and infra-red regions—of metallic gold in a finely divided condition, as found in gold-coloured glass, and as reduced from dilute aqueous solutions of its salts.